Int. Journal of Math. Analysis, Vol. 3, 2009, no. 27, 1331 - 1338

Neighborhood Properties of Generalized Ruscheweyh Type Analytic Functions

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Abstract. Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0,\ k \in \mathbb{N}\backslash\{1\},\ n \in \mathbb{N})$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. In this paper, the new subclasses $\mathcal{S}_{n,m}(\gamma,\lambda,\beta)$, $\mathcal{R}_{n,m}(\gamma,\lambda,\beta;\mu)$, $\mathcal{S}^{\alpha}_{n,m}(\gamma,\lambda,\beta)$ and $\mathcal{R}^{\alpha}_{n,m}(\gamma,\lambda,\beta;\mu)$ of $\mathcal{A}(n)$ are defined using generalized Ruscheweyh derivative and certain properties of neighborhoods for functions belonging to these classes are studied.

Mathematics Subject Classification: 30C45

Keywords: Univalent functions, neighborhoods, convex functions, starlike functions and Generalized Ruscheweyh operator

1. Introduction

Let $\mathcal{A}(n)$ denote the class of functions of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \quad (a_k \geq 0,\ k \in \mathbb{N}\backslash\{1\},\ n \in \mathbb{N}) \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$.
For any function $f(z) \in \mathcal{A}(n)$, $z \in \mathcal{U}$ and $\delta \geq 0$, we define

$$\mathcal{N}_{n,\delta}(f) = \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta \right\} \tag{1.2}$$

which is the (n, δ) - neighborhood of $f(z)$.
For $e(z) = z$, we see that

$$\mathcal{N}_{n,\delta}(e) = \left\{ g \in \mathcal{A}(n) : g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|b_k| \leq \delta \right\}. \tag{1.3}$$

The concept of neighborhoods was first introduced by Goodman [8] and then generalized by Ruscheweyh [10] .
In this paper, we discuss certain properties of (n, δ) - neighborhood for analytic functions of complex order in $\mathcal{U}$.
The subclass $\mathcal{S}_n^*(\gamma)$ [6] of $\mathcal{A}(n)$, is the class of functions of complex order γ satisfying,

$$\Re\left\{1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right\} > 0, \quad (z \in \mathcal{U},\ \gamma \in \mathbb{C}\backslash\{0\}). \tag{1.4}$$

The subclass $\mathcal{C}_n(\gamma)$ [6] of $\mathcal{A}(n)$, is the class of functions of complex order γ satisfying,

$$\Re\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > 0, \quad (z \in \mathcal{U},\ \gamma \in \mathbb{C}\backslash\{0\}). \tag{1.5}$$

The Hadamard product of two power series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

is defined as $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$.
In the present paper, using generalized Ruscheweyh derivative $\mathcal{D}_\lambda^m f$ [1], we define the new subclasses $\mathcal{S}_{n,m}(\gamma, \lambda, \beta)$, $\mathcal{R}_{n,m}(\gamma, \lambda, \beta; \mu)$ of $\mathcal{A}(n)$. The generalized Ruscheweyh derivative $\mathcal{D}_\lambda^m f$ is defined as follows.
For $f \in \mathcal{A}(n)$, $\lambda \geq 0$ and $m \in \mathbb{R}, m > -1$, we have

$$\mathcal{D}_\lambda^m f(z) = \frac{z}{(1-z)^{m+1}} * \mathcal{D}_\lambda f(z), \quad z \in \mathcal{U},$$

where $\mathcal{D}_\lambda f(z) = (1 - \lambda) f(z) + \lambda z f'(z)$.
If $f(z) \in \mathcal{A}(n)$, $f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k$, $z \in \mathcal{U}$, we obtain the power series expansion

of the form,

$$\mathcal{D}_\lambda^m f(z) = z - \sum_{k=n+1}^{\infty} [1 + (k-1)\lambda] \frac{(m+1)_{(k-1)}}{(1)_{(k-1)}} a_k z^k, \ \ z \in \mathcal{U}, \tag{1.6}$$

where $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & for\, n = 0 \\ a(a+1)(a+2)\cdots(a+n-1), & for\, n \in \mathbb{N}. \end{cases}$

Definition 1.1. *The subclass $\mathcal{S}_{n,m}(\gamma, \lambda, \beta)$ of $\mathcal{A}(n)$ is defined as the class of functions f such that*

$$\left| \frac{1}{\gamma} \left(\frac{z(\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)} - 1 \right) \right| < \beta, \tag{1.7}$$

where, $\gamma \in \mathbb{C}\backslash\{0\}$, $0 < \beta \le 1$, $m > -1$ and $z \in \mathcal{U}$.

Definition 1.2. *Let the subclass $\mathcal{R}_{n,m}(\gamma, \lambda, \beta; \mu)$ of $\mathcal{A}(n)$ is defined as the class of functions f such that*

$$\left| \frac{1}{\gamma} \left((1-\mu) \frac{\mathcal{D}_\lambda^m f(z)}{z} + \mu (\mathcal{D}_\lambda^m f(z))' - 1 \right) \right| < \beta, \tag{1.8}$$

where, $\gamma \in \mathbb{C}\backslash\{0\}$, $0 < \beta \le 1$, $m > -1$ and $z \in \mathcal{U}$.

The above two classes generalize the classes $\mathcal{S}_n(\gamma, \lambda, \beta)$, $\mathcal{R}_n(\gamma, \lambda, \beta; \mu)$ are studied by Murugusundaramoorthy and Srivasthava [9].

2. Neighborhoods for classes $\mathcal{S}_{n,m}(\gamma, \lambda, \beta)$ and $\mathcal{R}_{n,m}(\gamma, \lambda, \beta; \mu)$

In this section, we obtain inclusion relations involving $\mathcal{N}_{n,\delta}$ for functions in the classes $\mathcal{S}_{n,m}(\gamma, \lambda, \beta)$ and $\mathcal{R}_{n,m}(\gamma, \lambda, \beta; \mu)$.

Lemma 2.1. *A function $f(z) \in \mathcal{S}_{n,m}(\gamma, \lambda, \beta)$ if and only if*

$$\sum_{k=n+1}^{\infty} [1 + (k-1)\lambda] \frac{(m+1)_{k-1}}{(1)_{k-1}} (\beta|\gamma| + k - 1) a_k \ \le \ \beta|\gamma|. \tag{2.1}$$

Proof. Let $f(z) \in \mathcal{S}_{n,m}(\gamma, \lambda, \beta)$. Then, by (1.7) we can write,

$$\Re \left\{ \frac{(z\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m(z)} - 1 \right\} > -\beta|\gamma|, \quad (z \in \mathcal{U}). \tag{2.2}$$

Using (1.1) and (1.6), we have,

$$\Re \left\{ \frac{- \displaystyle\sum_{k=n+1}^{\infty} [1 + (k-1)\lambda] \frac{(m+1)_{k-1}}{(1)_{k-1}} (k-1) a_k z^k}{z - \displaystyle\sum_{k=n+1}^{\infty} [1 + (k-1)\lambda] \frac{(m+1)_{k-1}}{(1)_{k-1}} a_k z^k} \right\} > -\beta|\gamma|, \quad (z \in \mathcal{U}). \tag{2.3}$$

Letting $z \to 1$, through the real values, the inequality (2.3) yields the desired condition (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z|=1$, we obtain,

$$\begin{aligned}
\left|\frac{z(\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)}-1\right| &= \left|\frac{\sum\limits_{k=n+1}^{\infty}[1+(k-1)\lambda]\,\frac{(m+1)_{k-1}}{(1)_{k-1}}(k-1)a_k z^k}{z-\sum\limits_{k=n+1}^{\infty}[1+(k-1)\lambda]\,\frac{(m+1)_{k-1}}{(1)_{k-1}}a_k z^k}\right| \\
&\le \frac{\beta|\gamma|\left(1-\sum\limits_{k=n+1}^{\infty}[1+(k-1)\lambda]\,\frac{(m+1)_{k-1}}{(1)_{k-1}}(k-1)a_k\right)}{1-\sum\limits_{k=n+1}^{\infty}[1+(k-1)\lambda]\,\frac{(m+1)_{k-1}}{(1)_{k-1}}a_k} \\
&\le \beta|\gamma|.
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z)\in\mathcal{S}_{n,m}(\gamma,\lambda,\beta)$, which establishes the required result. $\square$

On similar lines, we have the following Lemma.

Lemma 2.2. *A function* $f(z)\in\mathcal{R}_{n,m}(\gamma,\lambda,\beta;\mu)$ *if and only if*

$$\sum_{k=n+1}^{\infty}[1+(k-1)\lambda]\,\frac{(m+1)_{k-1}}{(1)_{k-1}}\,[\mu(k-1)+1]\,a_k\le\beta|\gamma|. \tag{2.4}$$

Theorem 2.3. *If*

$$\delta=\frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)(1+n\lambda)\frac{(m+1)_n}{(1)_n}},\quad(|\gamma|<1), \tag{2.5}$$

then $\mathcal{S}_{n,m}(\gamma,\lambda,\beta)\subset\mathcal{N}_{n,\delta}(e)$.

Proof. Let $f(z)\in\mathcal{S}_{n,m}(\gamma,\lambda,\beta)$. By Lemma 2.1, we have,

$$(\beta|\gamma|+n)(1+n\lambda)\frac{(m+1)_n}{(1)_n}\sum_{k=n+1}^{\infty}a_k\le\beta|\gamma|.$$

which implies,

$$\sum_{k=n+1}^{\infty}a_k\le\frac{\beta|\gamma|}{(\beta|\gamma|+n)(1+n\lambda)\frac{(m+1)_n}{(1)_n}}. \tag{2.6}$$

Using (2.1) and(2.6), we have,

$$(1+n\lambda)\frac{(m+1)_n}{(1)_n}\sum_{k=n+1}^{\infty} ka_k \le \beta|\gamma| + (1-\beta|\gamma|)(1+n\lambda)\frac{(m+1)_n}{(1)_n}\sum_{k=n+1}^{\infty} a_k$$

$$\le \frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)(1+n\lambda)\frac{(m+1)_n}{(1)_n}} = \delta.$$

That is,

$$\sum_{k=n+1}^{\infty} ka_k \le \frac{(n+1)\beta|\gamma|}{(\beta|\gamma|+n)(1+n\lambda)\frac{(m+1)_n}{(1)_n}} = \delta.$$

Thus, by the definition given by (1.3), $f(z) \in \mathcal{N}_{n,\delta}(e)$, which completes the proof. $\square$

Theorem 2.4. *If*

$$\delta = \frac{(n+1)\beta|\gamma|}{(\mu n+1)(1+n\lambda)\frac{(m+1)_n}{(1)_n}}, \quad (|\gamma| < 1), \tag{2.7}$$

then $\mathcal{R}_{n,m}(\gamma,\lambda,\beta;\mu) \subset \mathcal{N}_{n,\delta}(e)$.

Proof. Let $f(z) \in \mathcal{R}_{n,m}(\gamma,\lambda,\beta;\mu)$. Then, by Lemma 2.2, we have,

$$(1+n\lambda)\frac{(m+1)_n}{(1)_n}(\mu n+1)\sum_{k=n+1}^{\infty} a_k \le \beta|\gamma|,$$

which gives the following coefficient inequality,

$$\sum_{k=n+1}^{\infty} a_k \le \frac{\beta|\gamma|}{(\mu n+1)(1+n\lambda)\frac{(m+1)_n}{(1)_n}}. \tag{2.8}$$

Using (2.4) and (2.8), we also have,

$$\mu(1+n\lambda)\frac{(m+1)_n}{(1)_n}\sum_{k=n+1}^{\infty} ka_k \le \beta|\gamma| + (\mu-1)(1+n\lambda)\frac{(m+1)_n}{(1)_n}\sum_{k=n+1}^{\infty} a_k$$

$$\le \beta|\gamma| + (\mu-1)(1+n\lambda)\frac{(m+1)_n}{(1)_n}\frac{\beta|\gamma|}{(\mu n+1)(1+n\lambda)\frac{(m+1)_n}{(1)_n}}.$$

That is,

$$\sum_{k=n+1}^{\infty} ka_k \le \frac{(n+1)\beta|\gamma|}{(\mu n+1)(1+n\lambda)\frac{(m+1)_n}{(1)_n}} = \delta.$$

Thus, by the definition given by (1.3), $f(z) \in \mathcal{N}_{n,\delta}(e)$, which completes the proof. $\square$

3. Neighborhoods for classes $\mathcal{S}^{\alpha}_{n,m}(\gamma,\lambda,\beta)$ and $\mathcal{R}^{\alpha}_{n,m}(\gamma,\lambda,\beta;\mu)$

In this section, we define the subclasses $\mathcal{S}^{\alpha}_{n,m}(\gamma,\lambda,\beta)$ and $\mathcal{R}^{\alpha}_{n,m}(\gamma,\lambda,\beta;\mu)$ of $\mathcal{A}(n)$ and neighborhoods of these classes are obtained.
For $0 \leq \alpha < 1$ and $z \in \mathcal{U}$, a function $f(z) \in \mathcal{S}^{\alpha}_{n,m}(\gamma,\lambda,\beta)$ if there exists a function $g(z) \in \mathcal{S}_{n,m}(\gamma,\lambda,\beta)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \alpha. \tag{3.1}$$

For $0 \leq \alpha < 1$ and $z \in \mathcal{U}$, a function $f(z) \in \mathcal{R}^{\alpha}_{n,m}(\gamma,\lambda,\beta;\mu)$ if there exists a function $g(z) \in \mathcal{R}_{n,m}(\gamma,\lambda,\beta;\mu)$ such that the inequality (3.1) holds true.

Theorem 3.1. *If* $g(z) \in \mathcal{S}_{n,m}(\gamma,\lambda,\beta)$ *and*

$$\alpha = 1 - \frac{(\beta|\gamma| + n)\,\delta\,(1 + n\lambda)\dfrac{(m+1)_n}{(1)_n}}{(n+1)\left[(\beta|\gamma| + n)(1 + n\lambda)\dfrac{(m+1)_n}{(1)_n} - \beta|\gamma|\right]}, \tag{3.2}$$

then $\mathcal{N}_{n,\delta}(g) \subset \mathcal{S}^{\alpha}_{n,m}(\gamma,\lambda,\beta)$.

Proof. Let $f(z) \in \mathcal{N}_{n,\delta}(g)$. Then,

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta, \tag{3.3}$$

which yields the coefficient inequality,

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1}, \quad (n \in \mathbb{N}). \tag{3.4}$$

Since $g(z) \in \mathcal{S}_{n,m}(\gamma,\lambda,\beta)$ by (2.6), we have,

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\beta|\gamma|}{(\beta|\gamma| + n)(1 + n\lambda)\dfrac{(m+1)_n}{(1)_n}}, \tag{3.5}$$

so that,

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\displaystyle\sum_{k=n+1}^{\infty}|a_k-b_k|}{1-\displaystyle\sum_{k=n+1}^{\infty}b_k}\\[2ex] &\leq \frac{\delta}{n+1}\,\frac{(\beta|\gamma|+n)(1+n\lambda)\dfrac{(m+1)_n}{(1)_n}}{\left[(\beta|\gamma|+n)(1+n\lambda)\dfrac{(m+1)_n}{(1)_n}-\beta|\gamma|\right]}\\[2ex] &= 1-\alpha.\end{aligned}$$

Thus, by definition, $f(z)\in\mathcal{S}^{\alpha}_{n,m}(\gamma,\lambda,\beta)$ for α given by (3.2), which establishes the desired result. $\square$

On similar lines, we can prove the following Theorem.

Theorem 3.2. *If* $g(z)\in\mathcal{R}_{n,m}(\gamma,\lambda,\beta;\mu)$ *and*

$$\alpha = 1-\frac{(\mu n+1)\,\delta\,(1+n\lambda)\dfrac{(m+1)_n}{(1)_n}}{(n+1)\left[(\mu n+1)(1+n\lambda)\dfrac{(m+1)_n}{(1)_n}-\beta|\gamma|\right]}, \tag{3.6}$$

then $\mathcal{N}_{n,\delta}(g)\subset\mathcal{R}^{\alpha}_{n,m}(\gamma,\lambda,\beta;\mu)$.

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Received: November, 2008